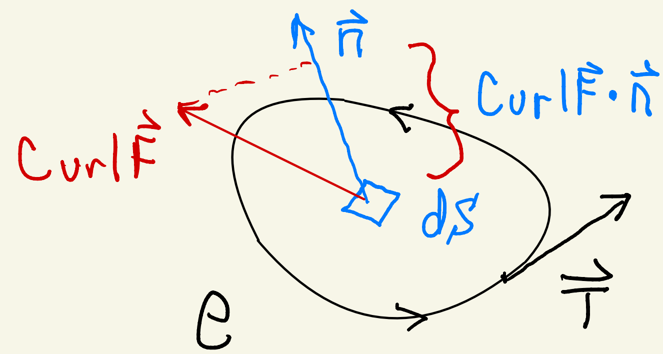
Meaning of Stokes Theorem:

Stokes theorem: $$\iint_S \operatorname{Curl}\vec{F}\cdot\vec{n}\,dS = \int_C \vec{F}\cdot\vec{T}\,ds$$

Flux of the Curl $\vec{F}$ thru S — line integral around boundary

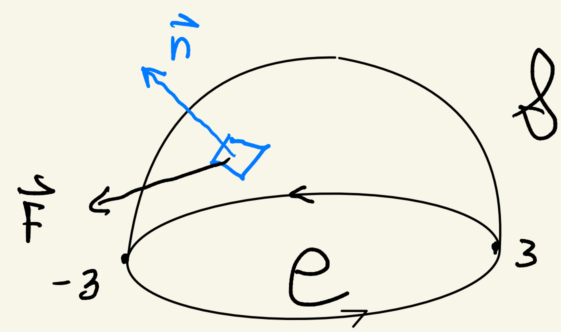


Example: Verify Stokes Theorem in case S = hemisphere

$$x^2+y^2+z^2=9, \quad z\geq 0$$

$$\vec{F} = y\,\underset{\sim}{i} - x\,\underset{\sim}{j}$$

Soln: The boundary Curve for the hemisphere is: $C = x^2+y^2=3$



- First we calculate RHS:

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C M\,dx + N\,dy = \int_0^{2\pi} 3y(-\sin t) - 3x\cos t\,dt$$

$3\sin t$ $\quad$ $-3\cos t$

$t=\theta$

$x=3\cos t \quad y=3\sin t$

$0\leq t\leq 2\pi$

$dx=-3\sin t\,dt$

$dy=3\cos t\,dt$

$$= 9\int_0^{2\pi} -\sin^2 t - \cos^2 t\,dt = -9\cdot 2\pi = \boxed{-18\pi}$$

- Now calculate LHS: $\iint\limits_{S} \operatorname{Curl}\vec{F}\cdot\vec{n}\, dS$

$$\operatorname{Curl}\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & 0 \end{vmatrix} = \underset{\sim}{k}(-1-1) = -2k$$

$$\vec{r}(x,y) = \overrightarrow{\left(x, y, \sqrt{9-x^2+y^2}\right)}$$

$$\vec{n} = \frac{x\underset{\sim}{i} + y\underset{\sim}{j} + zk}{\sqrt{x^2+y^2+z^2}} = \frac{x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}}{3}$$

because S is on sphere

$$\operatorname{Curl}\vec{F}\cdot\vec{n} = (-2\underset{\sim}{k})\cdot(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k})\frac{1}{3} = -\frac{2}{3}z$$

Spherical Coordinates: $u=\varphi$, $v=\theta$

$$\vec{r}(\varphi,\theta) = 3\overrightarrow{(\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)}$$

We have: $A = |\vec{r}_\varphi \times \vec{r}_\theta| = 3^2\sin\varphi$ (From map problem — or just compute)

$$\iint\limits_{S} \operatorname{Curl}\vec{F}\cdot\vec{n}\, dS = \iint\limits_{S} -\frac{2}{3}z\, dS = \int_0^{\pi/2}\int_0^{2\pi} -\frac{2}{3}\underbrace{z}_{3\cos\varphi}\, 3^2\sin\varphi\, d\theta\, d\varphi$$

$$= -\frac{2}{3}3^3(2\pi)\int_0^{\pi/2}\cos\varphi\sin\varphi\, d\varphi = -36\pi\left.\frac{\sin^2\varphi}{2}\right]_0^{\pi/2} = \boxed{-18\pi}$$

$u = \sin\varphi$

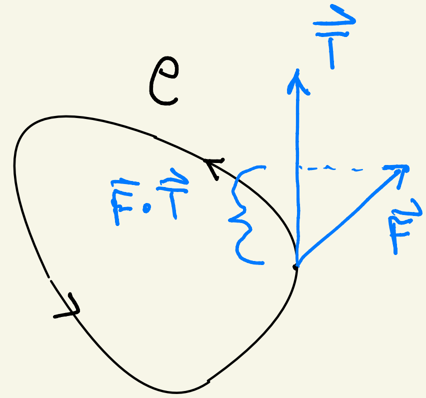
$du = \cos\varphi\, d\varphi$

Example 2 Use Stokes to obtain the correct interpretation of $\text{Curl}\vec{F}$ as "circulation per area in $\vec{F}$"

Soln: Let $\vec{F}$ be a vector field. Recall that

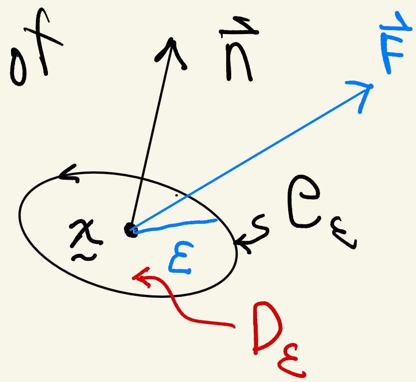
$\oint_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds$ is the "circulation in $\vec{F}$ around $\mathcal{C}$"

because it measures the component of $\vec{F}$ tangent to $\mathcal{C}$, weighted with arclength ds, and summed around the curve $\mathcal{C}$.



So... take a small disc D_ε of radius $\varepsilon > 0$, oriented with normal $\vec{n}$, placed at a point $\underset{\sim}{x} = (x, y, z)$.

Let $\mathcal{C}_\varepsilon$ be the boundary circle of D_ε, oriented by RHR with $\vec{n}$.



Now apply Stokes Theorem:

$$\iint_{D_\varepsilon} \underbrace{\text{Curl}\vec{F}\cdot\vec{n}\,dS}_{\text{We wonder what } \text{Curl}\vec{F}\cdot\vec{n} \text{ measures}} = \underbrace{\oint_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds}_{\text{We know this is Circulation in } \vec{F} \text{ around } \mathcal{C}}$$

Now for the trick: assuming $\vec{F}$ is smoothly varying, (say continuous derivatives) we know that as $\varepsilon \to 0$, the value of $\text{Curl}\,\vec{F}\cdot\vec{n}$ in D_ε is very close, ie. tends to it's value at the center, namely, $\text{Curl}\,\vec{F}\cdot\vec{n}\,(\underset{\sim}{x})$.

Thus we can approximate $\text{Curl}\,\vec{F}\cdot\vec{n}$ as constant, and pull it out of $\iint_{D_\varepsilon}$, only incuring a small error which will be negligible as $\varepsilon \to 0$.

$$\text{I.e. } \iint_{D_\varepsilon} \text{Curl}\,\vec{F}\cdot\vec{n}\, dS = \underbrace{\text{Curl}\,\vec{F}\cdot\vec{n}\,(\underset{\sim}{x})}_{\text{what we are trying to interpret}} \underbrace{\iint_{D_\varepsilon} dS}_{\text{Area of } D_\varepsilon \,=\, |D_\varepsilon| \,=\, \pi\varepsilon^2} + \underbrace{\text{error}}_{\text{smaller than } |D_\varepsilon|}$$

Thus from Stokes Theorem:

$$\iint_{D_\varepsilon} \text{Curl}\,\vec{F}\cdot\vec{n}\, dS = \text{Curl}\,\vec{F}\cdot\vec{n}\,(\underset{\sim}{x})\,|D_\varepsilon| + \text{error} = \oint_{C_\varepsilon} \vec{F}\cdot\vec{T}\, ds$$

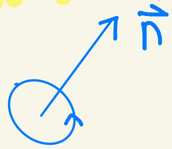
Divide thru by $|D_\varepsilon|$...

$$\text{Curl}\,\vec{F}\cdot\vec{n}\,(\underset{\sim}{x}) = \frac{1}{|D_\varepsilon|}\oint_{C_\varepsilon} \vec{F}\cdot\vec{T}\, ds - \underbrace{\frac{\text{error}}{|D_\varepsilon|}}_{\text{tends to zero as } \varepsilon \to 0}$$

Thus: $\operatorname{Curl}\vec{F}\cdot\vec{n}(\underset{\sim}{x}) = \lim_{\varepsilon \to 0} \underbrace{\frac{1}{|D_\varepsilon|} \oint_{C_\varepsilon} \vec{F}\cdot\vec{T}\, ds}_{\text{circulation per area}}$

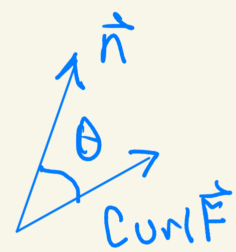
(Applies to any area D_ε oriented by normal $\vec{n}$)

Conclude: The value of $\operatorname{Curl}\vec{F}\cdot\vec{n}$ at a point $\underset{\sim}{x}$ measures the circulation per area in $\vec{F}$ around the axis $\vec{n}$.



Example ③ Around which axis $\vec{n}$ does $\vec{F}$ exhibit maximum circulation?

Ans: $\operatorname{Curl}\vec{F}\cdot\vec{n} = \|\operatorname{Curl}\vec{F}\|\,\|\vec{n}\|\cos\theta$



which is maximum when $\cos\theta = 1$, $\theta = 0$,

So the axis $\vec{n}$ around which $\vec{F}$ circulates most rapidly is $\vec{n} = \dfrac{\operatorname{Curl}\vec{F}}{\|\operatorname{Curl}\vec{F}\|}$, ie the direction of Curl $\vec{F}$!

Example 4 What does the length of $\text{Curl}\,\vec{F}$ measure?

Ans: The maximum circulation per area occurs around axis $\vec{n} = \dfrac{\text{Curl}\,\vec{F}}{\|\text{Curl}\,\vec{F}\|}$, the magnitude being

$$\text{Curl}\,\vec{F} \cdot \vec{n} = \text{Curl}\,\vec{F} \cdot \frac{\text{Curl}\,\vec{F}}{\|\text{Curl}\,\vec{F}\|} = \frac{\|\text{Curl}\,\vec{F}\|^2}{\|\text{Curl}\,\vec{F}\|} = \|\text{Curl}\,\vec{F}\|.$$

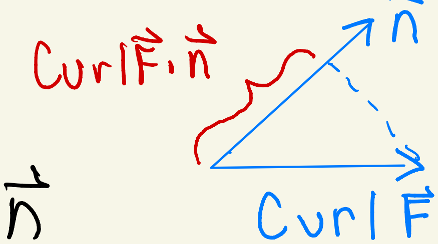
= maximum circulation per area

Conclude: $\text{Curl}\,\vec{F}$ gives the axis around which $\vec{F}$ is circulating most rapidly, and its length is the magnitude of maximum circulation!

Note: The component of $\text{Curl}\,\vec{F}$ in any unit direction $\vec{n}$ gives circulation per area around $\vec{n}$

$$\begin{pmatrix}\text{Circulation per} \\ \text{area around } \vec{n}\end{pmatrix} = \text{Curl}\,\vec{F} \cdot \vec{n} = \|\text{Curl}\,\vec{F}\| \cos\theta$$

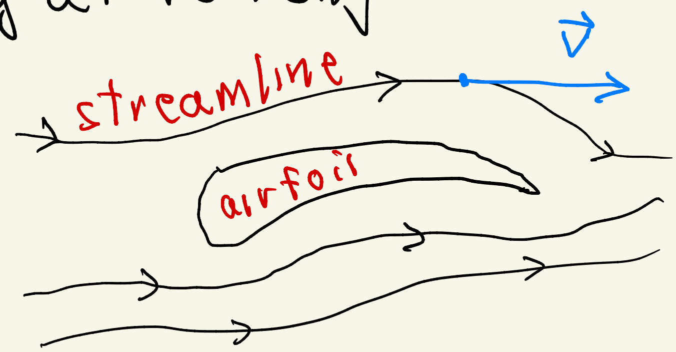
= Component of $\text{Curl}\,\vec{F}$ in direction of unit vector $\vec{n}$



Application: The Curl as "revolutions per second" = **frequency** in **fluid model** –

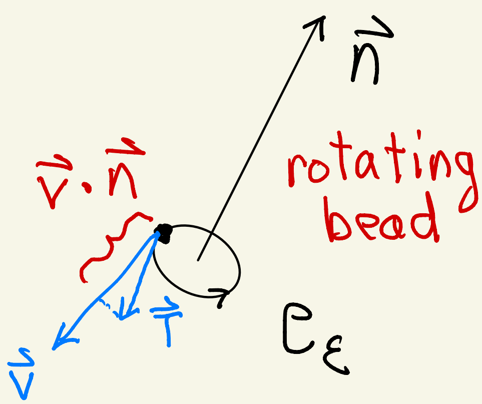
Assume $\vec{F} = \vec{v}$ = velocity in fluid model of a density $\delta(x,y,z)$ moving at velocity $\vec{v} = \vec{v}(x,y,t)$

I.e. a streamline $\vec{r}(t)$ is the curve taken by a fluid particle, and the velocity vector $\vec{v}(\underset{\sim}{x}) = \vec{r}\,'(t)$ for the streamline thru $\underset{\sim}{x} = (x,y,z)$



Q: What does Curl $\vec{v}$ measure at pt (x,y,z) ?

Ans: $\text{Curl}\,\vec{v}\cdot\vec{n} = 4\pi\omega$ where $\omega = \dfrac{\text{revolutions}}{\text{sec}}$ of a bead circulating on a circular wire C_ε oriented by $\vec{n}$ assuming $\vec{v}\cdot T$ gives the velocity (i.e., no friction, no loss of momentum) in limit $\varepsilon \to 0$.



Check: **Dimensions** $[v_x] = \frac{1}{L}[v] = \frac{1}{T}$

$$[\text{Curl}\,\vec{v}\cdot n] = \underset{\frac{1}{L}\,\frac{L}{T}}{[\text{Curl}\,\vec{v}]}\;\underset{1}{[\vec{n}]} = \frac{1}{T}$$

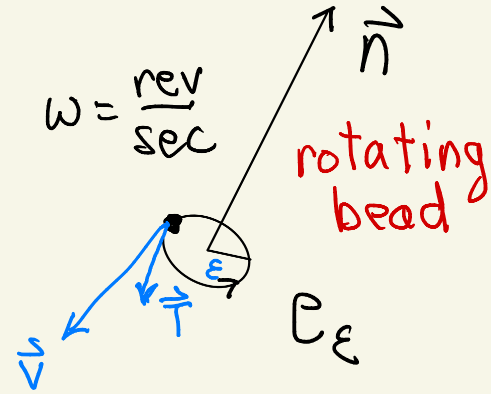
↖ dimension of frequency

$\left(\left[\frac{\vec{w}}{\|\vec{w}\|}\right] = \frac{[\vec{w}]}{[\|\vec{w}\|]} = 1 \Rightarrow \text{unit vectors are dimensionless} \right)$

Example ⑤: Verify $\operatorname{Curl}\vec{v}\cdot\vec{n} = 4\pi\omega$

Soln: $\operatorname{Curl}\vec{v}\cdot\vec{n} \underset{\substack{\uparrow \\ \text{Stokes}}}{\cong} \dfrac{\text{Circulation}}{\text{area}}$

$\omega = \dfrac{\text{rev}}{\text{sec}}$



$$= \underbrace{\frac{1}{\pi\varepsilon^2}}_{\text{area}} \int_{C_\varepsilon} \vec{v}\cdot\vec{T}\,ds$$

(The approximate equality $\cong$ becomes $=$ in limit $\varepsilon \to 0$)

But

$$\int_{C_\varepsilon} \underbrace{\vec{v}\cdot\vec{T}}\,ds \underset{ds=\varepsilon d\theta}{=} \int_0^{2\pi} \vec{v}\cdot\vec{T}\,\varepsilon\,d\theta$$

$\vec{v}\cdot\vec{T} = \dfrac{ds}{dt}$

$= \dfrac{\text{dist}}{\text{time}}$ of bead around C_ε

$$= 2\pi\varepsilon \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \vec{v}\cdot\vec{T}(\theta)\,d\theta}$$

$\bar{v}$ = average speed $\dfrac{ds}{dt}$

$$= 2\pi\varepsilon\bar{v}$$

Check:

$$\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{N\Delta t}\lim_{N\to\infty}\sum_{k=1}^{N} f(t_k)\,\Delta t_k = \lim_{N\to\infty} \underbrace{\frac{\sum_{k=1}^{N} f(t_k)}{N}}_{\text{average of } f}$$

Conclude: $\text{Curl}\,\vec{v}\cdot\vec{n} \cong \frac{1}{\cancel{\pi}\varepsilon^{\cancel{2}}}\, 2\cancel{\pi}\cancel{\varepsilon}\,\bar{v} = \frac{2}{\varepsilon}\,\bar{v}$

Now: at average speed $\bar{v}$, the bead makes one revolution around $2\pi\varepsilon$ = circumference of C_ε in time period T where

$$2\pi\varepsilon = \bar{v}\cdot T \quad\Rightarrow\quad T = \frac{2\pi\varepsilon}{\bar{v}}$$

$$\text{dist} = \frac{\text{dist}}{\text{time}}\cdot\text{time}$$

Thus: $\frac{\#\text{rev}}{\text{sec}} = \frac{1}{\text{time of one rev}} = \frac{1}{T} = \frac{\bar{v}}{2\pi\varepsilon} = \omega$

Conclude:

$$\omega = \frac{\bar{v}}{2\pi\varepsilon}, \qquad \text{Curl}\,\vec{v}\cdot\vec{n} = \frac{2}{\varepsilon}\,\bar{v}$$

$$\bar{v} = 2\pi\varepsilon\omega \quad\Rightarrow\quad \text{Curl}\,\vec{v}\cdot\vec{n} = \frac{2}{\cancel{\varepsilon}}\,2\pi\cancel{\varepsilon}\,\omega = 4\pi\omega \quad \checkmark$$

Summary: In the fluid model, the Curl $\vec{v}$ gives axis of maximal rotation of a bead constrained to move around a circular wire oriented $\perp$ Curl $\vec{v}$. The length $\|\text{Curl}\,\vec{v}\|$ then gives the maximal frequency of rotation, and Curl $\vec{v}\cdot\vec{n}$ gives frequency of rotation around axis $\vec{n}$.

In Fluid Mechanics: Curl $\vec{v}$ = vorticity. Vorticity plays a fundamental role in the theory of fluids.

Example (6) Assume a fluid is moving with velocity vector $\vec{F} = \vec{V} = x\underset{\sim}{i} + xyz\underset{\sim}{k}$ $\frac{m}{s}$ $\left(\frac{m}{s} = \frac{\text{meters}}{\text{sec}}\right)$

(1) Find axis of maximal rotation at $P = (1, -1, 2) = \underset{\sim}{x}_0$

(2) Find the maximal circulation/area

(3) Find the frequency ω and period T for a bead rotating with $\vec{v}$ around a circle of radius ε, center $\underset{\sim}{x}_0$, around axis $\vec{A} = \overrightarrow{(2, -1, 1)}$.

Soln (1)

$$\text{Curl}\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ x & 0 & xyz \end{vmatrix} = \underset{\sim}{i}(xz - 0) - \underset{\sim}{j}(yz - 0) + \underset{\sim}{k}(0 - 0)$$

$$= xz\underset{\sim}{i} - yz\underset{\sim}{j}$$

$$\text{Curl}\vec{F} = \overrightarrow{(xz, -yz, 0)}$$

@ $\underset{\sim}{x}_0 = (1, -1, 2)$, $\text{Curl}\vec{F} = \overrightarrow{(2, 2, 0)}$

$\therefore$ axis of max'al rotation $\vec{n} = \dfrac{\text{Curl}\vec{F}}{\|\text{Curl}\vec{F}\|} = \dfrac{\overrightarrow{(2,2,0)}}{\sqrt{4+4+0}} = \dfrac{\overrightarrow{(1,1,0)}}{\sqrt{2}}$

(2) Maximal Circulation per area $= \|\text{Curl}\vec{F}\| = \sqrt{8} = 2\sqrt{2}$ occurs around axis $\vec{n} = \dfrac{\overrightarrow{(1,1,0)}}{\sqrt{2}}$

(3) For $\vec{F} = \vec{V}$ $\frac{m}{s}$, $\text{Curl}\vec{v} = \overrightarrow{(xz, -yz, 0)}$, $\text{Curl}\vec{v} \cdot n = 4\pi\omega$

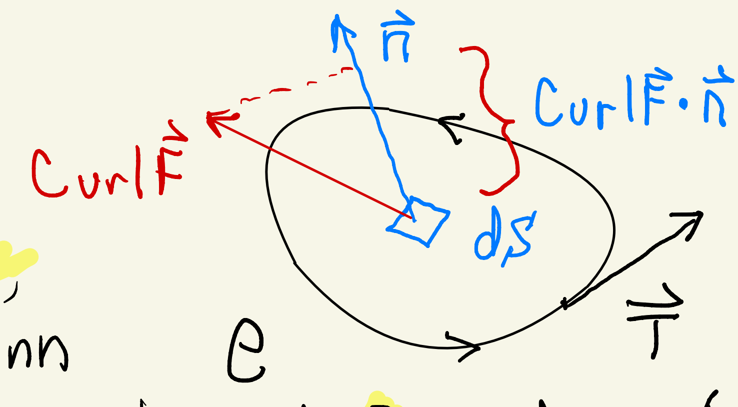
$$\omega = \frac{1}{4\pi}\text{Curl}\vec{v} \cdot n(\underset{\sim}{x}_0) = \frac{1}{4\pi}\overrightarrow{(2,2,0)} \cdot \frac{\vec{A}}{\|\vec{A}\|} = \frac{1}{4\pi}\frac{\overrightarrow{(2,2,0)} \cdot \overrightarrow{(2,-1,1)}}{\sqrt{4+1+1}} = \frac{2}{4\pi\sqrt{6}} = \frac{1}{\sqrt{6}\pi}$$

Ans: $\omega = \dfrac{1}{\sqrt{6}\pi}\dfrac{1}{s}$, $T = \dfrac{1}{\omega} = \sqrt{6}\pi$ Seconds

Why Stokes Theorem is true —

Stokes Theorem: $\iint_{S} \text{Curl}\vec{F}\cdot\vec{n}\, dS = \oint_{C} \vec{F}\cdot\vec{T}\, ds$

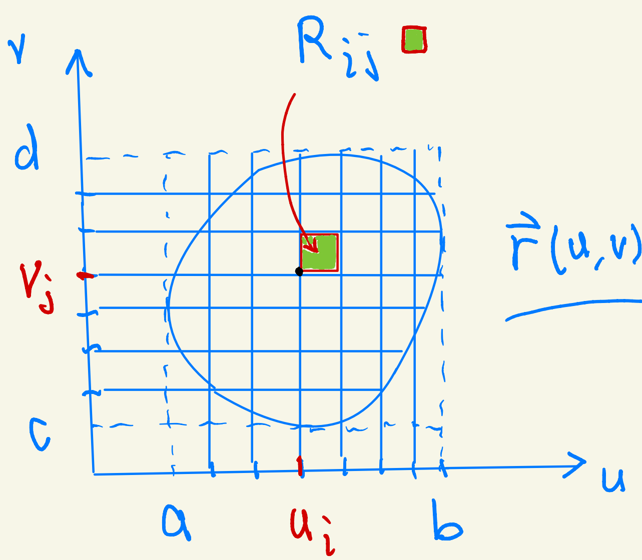
Q: Why is it true?

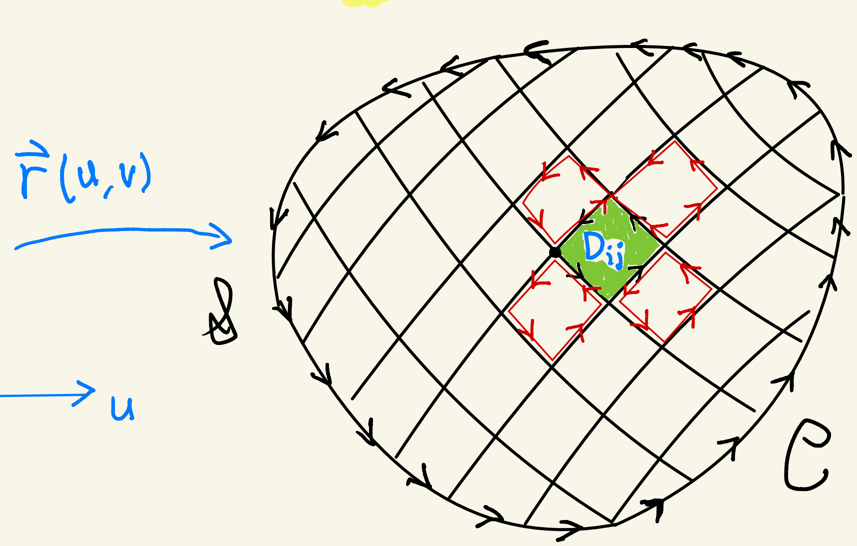


Ans: Because $\text{Curl}\vec{F}\cdot\vec{n}$ is the circulation per area, so when we write a Riemann sum for $\iint_{S} \text{Curl}\vec{F}\cdot\vec{n}\, dS$, each element D_{ij} reduces to a line integral around its boundary, and all the interior adjacent line integrals cancel out because they have opposite orientation.

I.e., $\iint_{S} \text{Curl}\vec{F}\cdot\vec{n}\, dS = \lim_{N\to\infty} \left| \sum_{ij} \iint_{D_{ij}} \text{Curl}\vec{F}\cdot\vec{n}\, dS = \sum_{ij} \int_{C_{ij}} \vec{F}\cdot\vec{T}\, ds \right|$

$= \int_{C} \vec{F}\cdot\vec{T}\, ds$



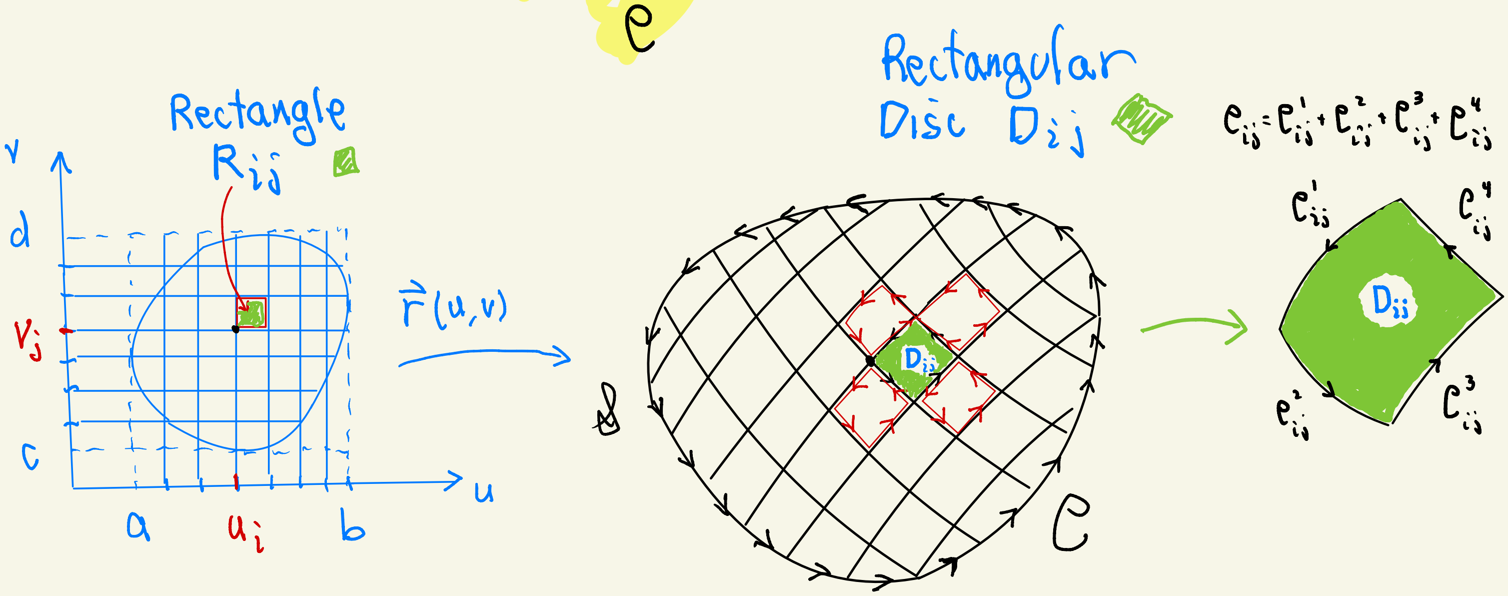


I.e., $\iint_{\mathcal{S}} \text{Curl}\vec{F}\cdot\vec{n}\, dS = \lim_{N\to\infty} \sum_{ij} \iint_{D_{ij}} \underbrace{\text{Curl}\vec{F}\cdot\vec{n}}_{\frac{1}{|D_{ij}|}\int_{\mathcal{C}_{ij}} \vec{F}\cdot\vec{T}\, ds}\ \underbrace{dS}_{|D_{ij}|}$

$= \lim_{N\to\infty} \sum_{ij} \int_{\mathcal{C}_{ij}} \vec{F}\cdot\vec{T}\, ds$

All adjacent line integrals cancel out leaving only boundary

$= \int_{\mathcal{C}} \vec{F}\cdot\vec{T}\, ds$



That is: all integrals on adjacent sides cancel out because they have opposite orientation, and the only line integrals left are the line integrals around outer boundary.

Note: This would be a proof, except we used Stokes Theorem to get $\iint_{\Delta S_{ij}} \text{Curl}\vec{F}\cdot\vec{n}\, dS = \int_{\mathcal{C}_{ij}} \vec{F}\cdot\vec{T}\, ds$, so we can't formally use that to prove Stokes Thm.

Conclude: The argument is circular in the sense that we assume Stokes Thm to interpret $\text{Curl}\,\vec{F}\cdot\vec{n}$ as circulation per area, then use this to prove Stokes Thm - Even so, this is the correct intuition for why Stokes Thm is TRUE!